Bits or Shots in Combat? August 13, 2013
The Generalized Deitchman Model of Guerrilla Warfare

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ABSTRACT:
We generalize Deitchman's guerrilla warfare model to account for trade-off between intelligence ('bits') and firepower ('shots'). Intelligent targeting leads to aimed fire; absence of intelligence leads to unaimed fire, dependent on targets' density. We propose a new Lanchester-type model that mixes aimed and unaimed fire, the balance between these being determined by quality of information. We derive the model's conserved quantity, and use it to analyze the trade-off between investments in intelligence and in firepower—for example, in counterinsurgency operations.

KEYWORDS: military; combat modeling; counterinsurgency

1 Introduction

Good intelligence is key for effective combat operations. If a shooter knows exactly the location and state of his targets, he can accurately target them with effective aimed fire. Absent such information, the shooter is essentially 'shooting in the dark'—utilizing unaimed fire whose effectiveness depends on the density of the targets. Arguably, such unaimed fire is less effective than aimed fire, and it may also result in substantial unintended collateral damage. At its simplest, the balance between aimed and unaimed fire boils down to the trade-off between situational awareness ('bits') and firepower ('shots').

Our purpose in this paper is to write down and analyze a simple, prototypical system of two coupled differential equations which mixes aimed and unaimed fire, in the sense of Lanchester's models. It is perhaps surprising that this has not been done before. A step towards it is Deitchman's guerrilla warfare model [1], an asymmetric variant of Lanchester's models [2] in which aimed fire from guerrilla forces is opposed by unaimed fire from conventional forces. Deitchman's model was extended by Schaffer [3], who used the model to suggest new military hardware. The idea of modeling the trade-off between firepower and intelligence in a Lanchester setting was first suggested by Schreiber [4], albeit in a somewhat different context. Schreiber's model uses a reciprocal switching function between aimed and unaimed fire, whereas ours has simple linear interpolation. The main resulting difference between the two approaches is in their behaviour as the battle is scaled up: higher engaged numbers result in

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14. ABSTRACT

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a shift towards aimed fire in the Schreiber model and towards unaimed fire in ours.

Our model captures the dynamics of a perennial problem of combat, which recurs in different contexts through the ages: finding the best trade-off between rate and accuracy of fire. This trade-off concerns both the optimal use of single weapons—archery, anti-aircraft fire, battleship gunnery and musketry from its inception to the modern day—and finding the correct weapons mix, such as SAMs and flak against aircraft, or depth charges and torpedos against submarines. After an initial analysis of the model we specialize, for simplicity and without much loss of generality, to the situation in which only one side has this mix of fire and thus faces the trade-off problem, while the other can aim all of its fire. Although our results are applicable wherever the rate/accuracy trade-off problem bites, we choose to illustrate it in the context of counterinsurgency (COIN) operations.

We assume that in COIN war it is straightforward for the insurgents to identify state forces, so that all insurgent fire is aimed. In fact we could equally have made it unaimed; the point is that only the state, with the problem of distinguishing insurgents from civilians, faces the bits *vs* shots trade-off problem. Assuming that the resources for both capabilities are derived from the same pot (*e.g.* defense budget) the question is how to allocate the resources between them. Typically, in COIN settings, the state forces are confronted by relatively small armed groups, diffused in the population, which are ill-equipped and poorly trained. In terms of physical net assessment, insurgents are no match for state forces, at least not in the early stages of the insurgency. For example, in September 2011, the estimated number of insurgents in Syria was around 10,000 people [5], while the Syrian armed forces (active, reserve and paramilitary personnel) were estimated at over 700,000 soldiers [6]. The key advantage of the insurgents is their elusiveness and invisibility while blended into the civilian population, which make it difficult for state forces to identify insurgent targets and execute effective COIN operations. Thus, while intelligence is a key component in any conflict situation, it is critical in COIN operations. The problem of the state is how to divide limited COIN resources between gathering information about the insurgents and accumulating firepower that can effectively engage them.

There have been many attempts to model insurgencies as dynamical systems. Descriptive models have addressed the effect of civilian collateral casualties generated by the state [7] and by the insurgents [8] on public response and, consequently, on the fate of the insurgency, the impact of collective memory on popular behavior towards the state and the insurgents [9], and the spatial dynamics of such conflicts [10]. Berman *et al.* model COIN as a three-way contest between violent insurgents, a state seeking to minimize violence, and civilians deciding whether to share information with the state [11]. A related paper compares two possible COIN tactics —'fire' (high violence) and 'water' (low violence)—using optimal control techniques [12]. Bohorquez *et al.* reveal, in an empirical study, some unique patterns regarding the size and timing distributions of insurgencies [13], which may be explained by

notions of coalescence and fragmentation of insurgent groups. The dynamical problems of intelligence collection itself are treated by Kaplan and collaborators [14], while Schaffer has introduced an updated model which contrasts 21st century insurgency with Vietnam [15]. For recent overviews of the literature on mathematical modeling of intelligence and warfare see [16] and [17], respectively. In this note, however, we do not attempt to deal with dynamical non-physical variables such as psychological and social effects in COIN. Our intention is to construct and analyze no more than a homogeneous model of attrition, with all of the simplifications that this implies. In Epstein's categorization of reasons to model [18], ours is to illustrate the core dynamics of the trade-off in combat attrition between two parameters: rate of fire, and intelligence. Important questions such as how popular opinion shifts over time, and how it impacts upon the evolution and outcome of a conflict, are dealt with in [7, 8, 9, 10, 11, 12].

In the next section we extend Deitchman's classical model for the case of partial intelligence (on both sides) and obtain the conserved quantity. In the rest of the paper we focus on the asymmetrical COIN situation. In Section 4 we discuss the trade-off between bits and shots when cost information is unavailable. In Section 5 we present two constrained optimization models when the costs of intelligence and firepower are known and the budget is either constrained or is to be minimized. We show that the two optimization problems produce the same optimal solution. Summary and conclusions are presented in Section 6.

2   The Generalized Deitchman Model

The core of the generalized model is a pair of parameters that interpolate aimed and unaimed fire, representing the intelligence levels—hereafter called 'intel'—of the two forces. The values of these parameter range between 0 (no intel—shooting in the dark, unaimed fire) and 1 (perfect intel—all fire is aimed). In this section we obtain and interpret the conserved quantity of such a mixed engagement.

Let the positive real variables $B(t)$ and $R(t)$ represent the sizes of the Blue and Red forces respectively, and β and ρ denote their per-unit aimed-fire hit-rates, so that the Lanchester aimed-fire model is

$$\dot{B} = -\rho R \,, \qquad \dot{R} = -\beta B \,, \tag{1}$$

where dots denote time-derivatives. Famously this system conserves $\beta B^2 - \rho R^2$, resulting in Lanchester's 'square law'. The Deitchman guerrilla model [1] mixes aimed fire by Red with unaimed fire by Blue, so that

$$\begin{aligned} \dot{B} &= -\rho R, \\ \dot{R} &= -\beta \frac{BR}{N_R}. \end{aligned} \tag{2}$$

Note that rather than introduce a single parameter for unaimed fire (which must necessarily have different dimensions than that for aimed fire) we retain β but introduce a new fixed parameter N_R, with the same dimensions as R and B, which parametrizes the density effect of unaimed fire. The Deitchman model conserves $\frac{1}{2}\beta B^2 - \rho N_R R$, so that Blue's fighting strength is square-law while Red's is linear-law, with Blue suffering a further disadvantage from the factor of $1/2$.

Our generalized Deitchman model introduces intel parameters μ and ν for Blue and Red, and is

$$
\begin{aligned}
\dot{B} &= -\rho R(\nu + (1-\nu)B/N_B), \\
\dot{R} &= -\beta B(\mu + (1-\mu)R/N_R).
\end{aligned}
\tag{3}
$$

Notice that Deitchman's model is obtained when $\mu = 0$ and $\nu = 1$.

Analogously to elementary Lanchester theory, we compute the conserved quantity Q for this system by dividing one equation by the other:

$$
\frac{dB}{dR} = \frac{\rho R(\nu + (1-\nu)B/N_B)}{\beta B(\mu + (1-\mu)R/N_R)}.
\tag{4}
$$

Separating variables and computing partial fractions we obtain the relationship

$$
\frac{\beta N_B}{(1-\nu)}\left(1 - \frac{1}{1 + \frac{1-\nu}{\nu}\frac{B}{N_B}}\right) dB = \frac{\rho N_R}{(1-\mu)}\left(1 - \frac{1}{1 + \frac{1-\mu}{\mu}\frac{R}{N_R}}\right) dR
\tag{5}
$$

between differentials, and then, by integrating, we find that the quantity

$$
Q := \frac{\beta N_B}{(1-\nu)}\left(B - \frac{\nu N_B}{1-\nu}\log\left(1 + \frac{1-\nu}{\nu}\frac{B}{N_B}\right)\right) - \frac{\rho N_R}{(1-\mu)}\left(R - \frac{\mu N_R}{1-\mu}\log\left(1 + \frac{1-\mu}{\mu}\frac{R}{N_R}\right)\right)
\tag{6}
$$

is constant throughout the battle. This expression interpolates between Lanchester linear and square laws, just as it should: in the $\mu, \nu \to 0$ limit the second, logarithmic term in each bracket vanishes and we have $Q = \beta N_B B - \rho N_R R$, the linear law, while in the $\mu, \nu \to 1$ limit, and after expanding the logarithm as a Taylor series, we have $Q = \frac{1}{2}\beta B^2 - \frac{1}{2}\rho R^2$, the square law. The mixed limit $\nu \to 1$, $\mu \to 0$ gives the conserved quantity $\frac{1}{2}\beta B^2 - \rho N_R R$ of the Deitchman model. We discuss these limits in more detail in the next section.

Blue's goal is to maximize Q over B, β and μ, Red's to minimize it over R, ρ and ν. In the Lanchester limits this is simple, since Q takes the form $Q = f(N_B, B, \beta) - f(N_R, R, \rho)$. So Blue (say) seeks to maximize f, independent of Red's choices, and an increase in its forces B by a factor k is equivalent to an increase in its hit-rate β by a factor k for the Linear Law and k^2 for the Square Law. In our model, however, the situation is more complex, for now $Q = f(N_B, B, \beta, \nu) - f(N_R, R, \rho, \mu)$, and Blue's and Red's optimal strategies are no longer independent, because ν is chosen by Red and μ by Blue. So Blue has to maximize Q given that Red is trying to minimize it, and vice versa. This moves us into the fascinating territory

of diffferential games [19]—but a full minimax analysis rapidly becomes unwieldy and unilluminating (in part because there is no canonical cost function), and is certainly beyond the scope of this note. Rather we proceed in the next section with a fuller analysis of the situation in which one side is able to aim all of its fire. The analysis is quite general, but is presented in the context of COIN.

3   COIN Model

Consider a COIN situation where $B = G$ are the state ('Governmant') forces and $R = I$ are the 'Insurgents', who are embedded in a civilian population of size P. The insurgents have perfect situational awareness regarding the state forces and therefore can utilize effective aimed fire, $\nu = 1$. The state forces, on the other hand, who do not have this perfect awareness, have to utilize a fraction $(1-\mu)$ of their firepower for unaimed engagement, where only a fraction $\frac{I}{P}$ of this firepower is effective. If the hit rates are α and γ for the insurgents and state, respectively, then (3) becomes

$$
\begin{aligned}
\dot{G} &= -\alpha I \\
\dot{I} &= -\gamma G \left(\mu + (1-\mu)\frac{I}{P} \right),
\end{aligned}
\tag{7}
$$

and the parity condition, for a fully-annihilating endgame at which $G = I = 0$, is

$$
\gamma G_0^2 = \frac{2\alpha P}{1-\mu} \left[I_0 - \frac{\mu P}{1-\mu} \log \left(1 + \frac{1-\mu}{\mu}\frac{I_0}{P} \right) \right]
\tag{8}
$$

where G_0 and I_0 are the initial force sizes of the state and the insurgency respectively.

We observe immediately from (8) that, in terms of the trade-off between hit-rate and force size, the state fights a Lanchester square law war: its fighting strength, the left-hand side of (8), is γG_0^2.

Defining $\kappa := \alpha/\gamma$, $y := G_0/P$, $x := I_0/P$ and $z := \frac{1-\mu}{\mu}x$ we obtain the parity condition in the simpler form

$$
y^2 = \frac{2\kappa\mu}{(1-\mu)^2} \left[z - \log(1+z) \right].
\tag{9}
$$

This has the expected limits. When z is small, which occurs if $\mu \simeq 1$ (mostly aimed fire) or $I \ll P$, one uses the Taylor series $\log(1+z) = z - z^2/2 + \dots$ to obtain

$$
y^2 \simeq \frac{\kappa\mu}{(1-\mu)^2} z^2 = \frac{\kappa}{\mu} x^2,
\tag{10}
$$

which results in a generalized square law

$$
\left(\frac{G_0}{I_0} \right)^2 \simeq \frac{\kappa}{\mu}.
\tag{11}
$$

At the other extreme, when $\mu \to 0$ we have $z \to \infty$ and $\log(1+z)/z \to 0$. The right-hand side of (9) then approaches $2\kappa x$, just as in the Deitchman, $\mu = 0$ case.

In Figure 1 we plot the phase portrait with direction field and parity curve. Under the generalized square law (11), the parity curve is linear, so that for a clear departure from linearity z must not be too small. Even if the insurgency is concentrated in a sparsely populated area, for example $P = 3I_0$ (that is, $x = 1/3$), we only see departure from linearity at very small values of μ. Figure 1 uses $x \le 1/3$, $\kappa = 1$, $\mu = 0.01$. For $\mu > 0.1$ the parity curve is practically indistinguishable from a straight line.

With limited endurance, where the state tolerates attrition up to $G_0 - \overline{G}$ and the insurgents surrender when their attrition reaches $I_0 - \overline{I}$ (where $\overline{G}, \overline{I} > 0$), the parity condition becomes

$$\frac{\gamma}{2}[G_0^2 - \overline{G}^2] = \frac{\alpha P}{1 - \mu}\left(I_0 - \overline{I} - \frac{\mu P}{1 - \mu}\log\left[\frac{1 + \frac{1-\mu}{\mu P}I_0}{1 + \frac{1-\mu}{\mu P}\overline{I}}\right]\right). \tag{12}$$
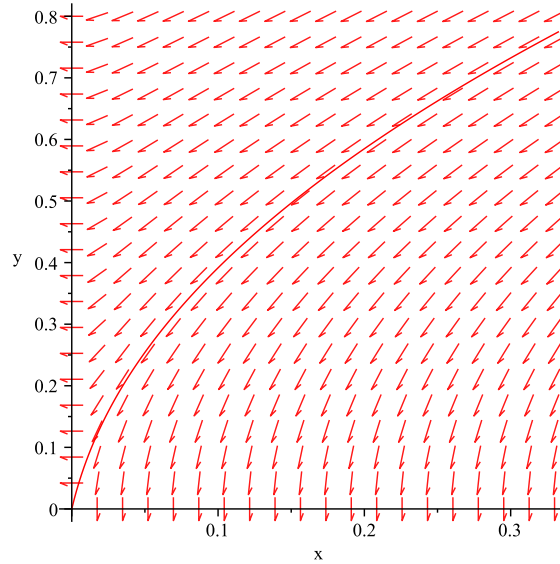

Figure 1: Phase portrait of generalized Deitchman model for $\kappa = 1$, $\mu = 0.01$. The parity curve separates the two forces' victory regimes.

4 Firepower-Intel Trade-off in COIN

In many modes of combat there is natural trade-off between firepower and intelligence that is manifested in fire rate: shoot now or wait for more accurate targeting information? Waiting for better intelligence (higher μ) results in larger inter-firing time and therefore lower γ. This trade-off is significant in particular in COIN situations because of the high cost of collateral damage.

Recall (as noted above) that in a square-law fight a proportionate improvement in numbers

is twice as valuable as the same proportionate improvement in unit hit-rate [2]. We formalize this with the logarithmic derivative,

$$d_\lambda := d(\log F)/d(\log \lambda) = \frac{\lambda}{F}\frac{dF}{d\lambda}, \tag{13}$$

where λ is some parameter and we take F to be the ratio of fighting strengths for the model, here

$$F := \frac{\frac{1}{2}\gamma G_0^2}{\frac{\alpha P^2 \mu}{(1-\mu)^2}[z - \log(1+z)]}, \tag{14}$$

with $z = \frac{1-\mu}{\mu}\frac{I_0}{P} = \frac{1-\mu}{\mu}x$ as before. Then we find immediately that $d_\gamma = 1$, $d_{G_0} = 2$, $d_\alpha = -1$: increasing numbers gives twice the improvement of increasing hit-rate or reducing vulnerability, the standard square law result.

For intel, though, the logarithmic derivative is not appropriate, essentially because $\gamma \in (0, \infty)$ whereas $\mu \in [0, 1]$. If instead we define a simple $D_\mu := \frac{d}{d\mu}\log F$, we find

$$D_\mu = \frac{1}{\mu}\left(1 - \frac{x}{\mu}\left[\frac{2}{z} - \frac{z}{(z+1)(z - \log(1+z))}\right]\right). \tag{15}$$

How does D_μ behave? As $\mu \to 1$, $D_\mu \to \frac{1}{\mu}(1 - \frac{2x}{3})$. As $\mu \to 0$ we find that D_μ has a logarithmic divergence; its behavior is $D_\mu \sim \frac{\log z}{x} - \frac{x+1}{x} + \mathcal{O}(\log z/z)$.

In Figure 2a we plot $D_\mu(\mu)$ for a representative x, here $x = 1/3$ as before. The crucial value is that at which $D_\mu = 1$, since it is here that crossover with $d_\gamma \equiv 1$ occurs. In Fig. 2a this critical μ_c is approximately 0.7. In Figure 2b we plot μ_c as a function of $1/x$ for integer values 1 through 20.
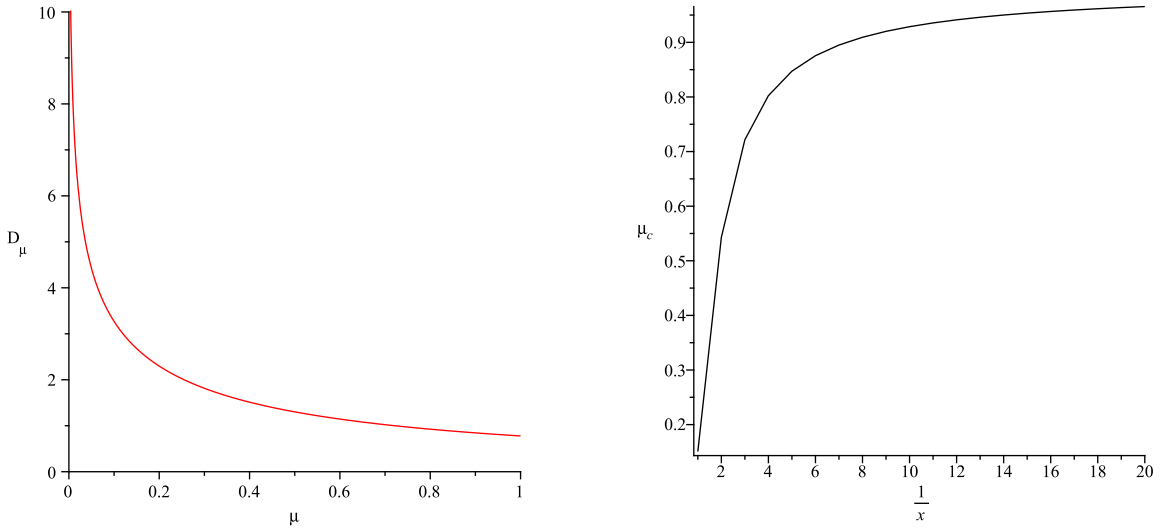

Figure 2: Plots of (a) D_μ against μ for $x = 1/3$, (b) critical values μ_c, at which $D_{\mu_c} = 1$, for $1/x = 1, \ldots, 20$.

So intel, which enables a small additional percentage of fire to be targeted accurately, is of greater value than an equivalent proportional increase in γ or G^2 provided $\mu < \mu_c$. The

operational lesson is that when intel is poor, it is better to turn a percentage of your untargeted fire into intelligent fire than it is to increase hit-rate by the same percentage. When intel is already high the opposite is true. This crossover happens at high values of μ, with $\mu_c \to 1$ as $x \to 0$: for example, if $x = 1/3$ then $\mu_c \simeq 0.72$. That is, if two-thirds of your untargeted fire goes astray, then more bits are better than more shots until nearly three-quarters of your fire is intelligent.

5 The Cost of COIN

The state wishes to reduce the insurgency, and the question is how to achieve the correct balance between intel and firepower efforts. As we noted above, our model is fundamentally Lanchester-like: it already combines lethality γ and numbers G_0, and thereby the trade-off between them, in the form γG_0^2. So we suppose now that the cost of COIN operations is linear in both efforts, the combined firepower γG_0^2 and intel μ. That is,

$$C(\text{COIN}) = c_1 \gamma G_0^2 + c_2 \mu \,, \tag{16}$$

and the trade-off between γ and μ is investigated by holding G_0 fixed. Such a cost function might naturally be extended to more general monotonic functions of γ, G_0 and μ, but we do not consider this here.

We consider two optimization problems. First, we minimize the total cost of conducting COIN operations subject to the constraint that the state does not lose the conflict, initially for a campaign of annihilation and then when there is limited loss toleration on both sides. Second, we maximize the force advantage (left-hand side minus right-hand of (8) when the forces are not at parity) subject to a fixed budget constraint.

5.1 Minimizing Cost

To simplify the model we first assume a full-annihilation case where $\overline{G} = \overline{I} = 0$. The objective now is to minimize $c_1 \gamma G_0^2 + c_2 \mu$ subject to the parity condition in (8). Substituting γG_0^2 computed from (8) in the cost function above we have

$$C(\mu) = c_2 \mu + c_1 \frac{2\alpha P}{(1-\mu)} \left[I_0 - \frac{\mu P}{1-\mu} \log \left(1 + \frac{1-\mu}{\mu} \frac{I_0}{P} \right) \right]. \tag{17}$$

Recall that $x = \frac{I_0}{P}$ is the relative initial inaccuracy of unintelligent fire. Define $r = \frac{c_2}{c_1 \alpha I_0^2}$, the intel-to-firepower cost ratio divided by (twice) the insurgent firepower at square law parity. Then

$$C(\mu) = 2\alpha c_1 P^2 f_{x,r}(\mu) \,, \tag{18}$$

where

$$f_{x,r}(\mu) = \frac{\mu r x^2}{2} + \frac{x}{(1-\mu)} - \frac{\mu}{(1-\mu)^2} \log\left(1 + \frac{1-\mu}{\mu}x\right). \tag{19}$$

Our task is to find the value of μ (in the interval $0 \leq \mu \leq 1$) which minimizes $f_{x,r}(\mu)$, for $0 < x < 1$ and $r > 0$.

First, note that $f_{x,r}(\mu) \to x$ and $f'_{x,r}(\mu) \to -\infty$ as $\mu \to 0$. Thus the minimum is always at a strictly-positive value of μ: however expensive intel may be, it is always best to have at least a little of it. (This might seem somewhat counterintuitive since, if $x = 1$, targeted and untargeted fire are equivalent initially, making any spend on intel initially wasteful. But note that, even if $x = I_0/P = 1$ initially, I/P will become less than one during the battle and will become small towards its end, when intel will become crucial.)

At the other end of the interval $f_{x,r}(1) = \frac{x^2(r+1)}{2}$ and $f'_{x,r}(1) = \frac{x^3}{3} + \frac{x^2(r-1)}{2}$ (via a series expansion of the logarithm and a little algebra). Thus (since we observe that f is either monotone or unimodal) the minimum cost is found at the end of the interval, $\mu = 1$, when $r \leq 1 - \frac{2x}{3}$. That is, other things equal, if intelligence is cheap enough then it is always better to acquire it in full capability. Since $x \leq 1$ necessarily, if $r \leq 1/3$ then this applies regardless of the signature value x.

For $r > 1 - \frac{2x}{3}$ the minimum is interior, $0 < \mu_{\min} < 1$, and our task is to understand how its location varies with x and r. An analytic solution is not illuminating. Rather we begin in Figure 3 by plotting a typical curve, here for $x = 1/3$, $r = 5$.
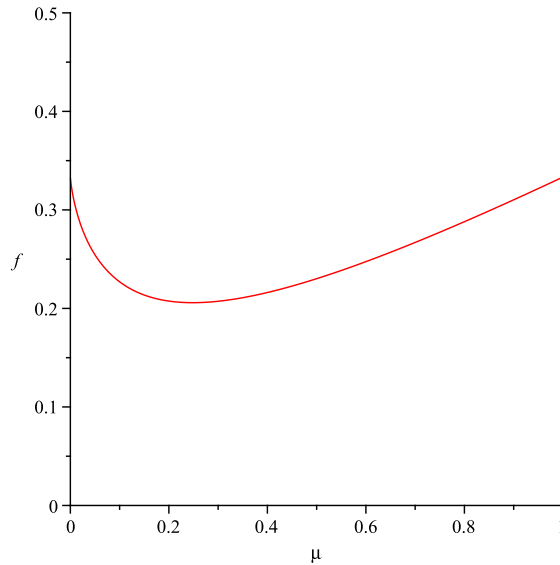

Figure 3: Scaled cost function $f(\mu)$
plotted against μ for $x = 1/3, r = 5$.

The minimum, here at about $\mu = 0.25$, decreases with increasing x or r. In Figure 4a, we generalize Fig. 3 to give a plot of $\mu_{\min}$, still with $x = 1/3$, for integer values of r from 1 to 10.

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Note that for $r = 1$ this minimum occurs at about 0.86; we saw above that when $r \leq 7/9$ (that is, $r = 1 - 2x/3$ with $x = 1/3$) the minimum reaches $\mu = 1$. The value at $r = 5$ is the $\mu_{\min}$ of Fig. 3.

In Figure 4b we generalize further to a plot of $\mu_{\min}$ as a function of r (again for integers from 1 to 10) and $1/x$ (also integers from 1 to 10). Fig. 4a is the section at $1/x = 3$. All calculations were performed using Maple 14.
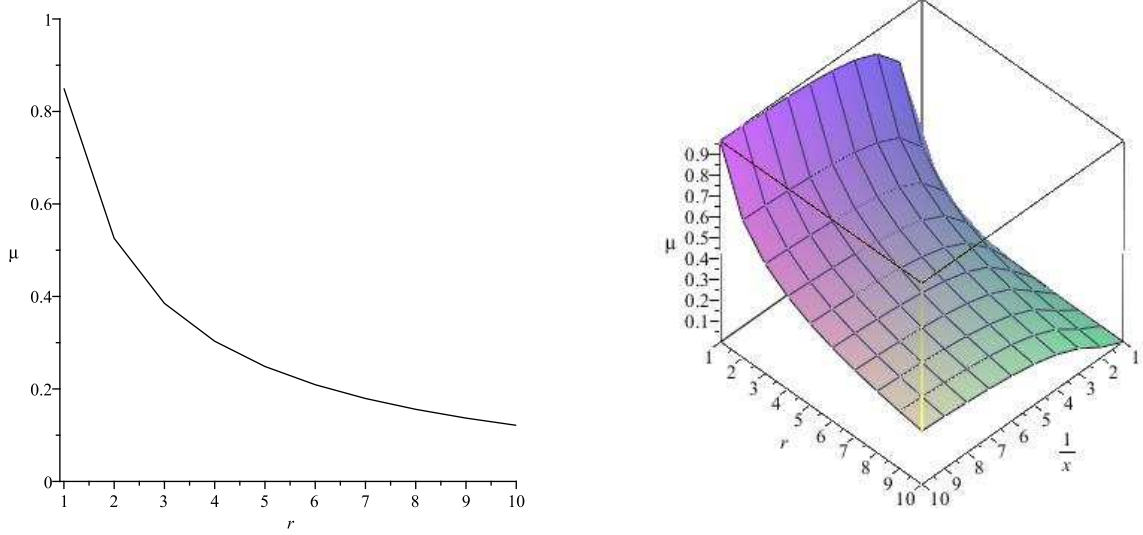

Figure 4: Plots of $\mu_{\min}$ (a) as a function of r for $x = 1/3$, (b) as a function of r and $1/x$.

5.2 Limited endurance

Next assume that each side has limited endurance, which results in the parity condition given in (12). Defining $\overline{x} = \frac{\overline{I}}{P}$ and $\overline{r} = r\frac{1-\overline{G}^2/G_0^2}{1-\overline{I}^2/I_0^2}$ we obtain from (12) and (16) that

$$C(\mu) = \frac{2\alpha c_1 P^2}{1 - \overline{G}^2/G_0^2} \left(f_{x,\overline{r}}(\mu) - f_{\overline{x},\overline{r}}(\mu) \right). \tag{20}$$

If we assume that each side has the same tolerated proportion of losses then $\overline{r} = r$ and the analysis of (20) becomes more tractable. First, we observe that $\mu = 1$ minimizes (20) when $r \leq 1 - \frac{2}{3}\frac{x^2+\overline{x}x+\overline{x}^2}{x+\overline{x}}$, a lower threshold than in the case of unlimited endurance where $\overline{x} = 0$. This means that limited loss toleration shifts the balance towards heavier weight on firepower versus intel.

For an alternate scenario, suppose that $\overline{I} = 0$ (the insurgency continues until its annihilation) but the state has limited loss toleration. Then $\mu = 1$ minimzes (20) whenever $\overline{r} \leq 1 - 2x/3$, and the cost range for which full intel is optimal increases with the state's inability to tolerate losses.

When $\bar{x} \to x$ (almost no toleration of losses), then, setting $\bar{x} = (1 - \epsilon)x$, we have

$$f_{x,r}(\mu) - f_{\bar{x},r}(\mu) \simeq \epsilon x^2 \left(\mu r + \frac{1}{\mu + (1 - \mu)x} \right) \tag{21}$$

and

$$\mu_{\min} = \begin{cases} 1 & r < 1 - x \\ \frac{1}{\sqrt{r(1-x)}} - \frac{x}{1-x} & 1 - x \leq r \leq \frac{1-x}{x^2} \\ 0 & r > \frac{1-x}{x^2}. \end{cases} \tag{22}$$

At $x = 1$ the solution is $\mu_{\min} = 0$: there is no initial difference between the effectiveness of aimed and of unaimed fire, and thus no value in intel.

Thus, whatever the loss toleration, for imperfectly-targeted fire ($x < 1$) if intel is cheap enough (that is, if r is low enough) then full intel is optimal. As loss toleration falls, this 'cheap enough' threshold also falls, from $r = 1 - \frac{2}{3}x$ in a war of annihilation to $r = 1 - x$ for minimal toleration of losses. If the state's unaimed fire is very poorly targeted ($x \ll 1$) then the threshold is approximately $r = 1$, independent of loss toleration.

When full intel is not optimal, we can compare limited loss toleration with the annihilating case. Figure 5a shows the plot of $\mu_{\min}$ against r, still with $x = 1/3$ and analogous to the ($\bar{x} = 0$) annihilating case in Fig. 4a, but now with $\bar{x} = 1/5$. Figure 5b shows the difference in $\mu_{\min}$ between the limited loss toleration case and the annihilating case—that is, the curve of Fig. 4a minus that of Fig. 5a. As was mentioned above, we observe that, compared to annihilation, limited loss toleration results in lower optimal values for intel.
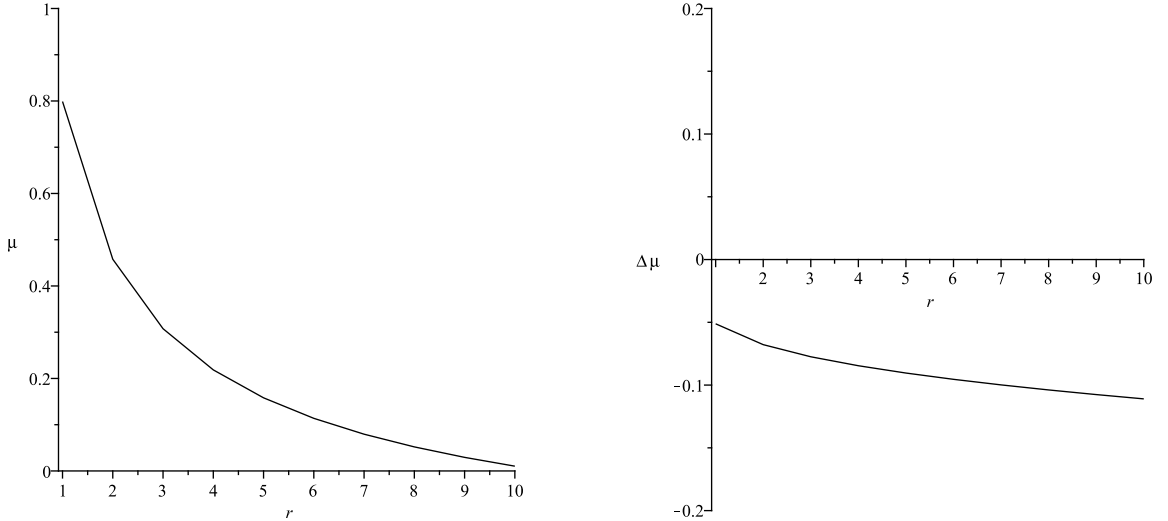

Figure 5: (a) plot of $\mu_{\min}$ for limited losses $\bar{x} = 1/5$, (b) difference $\Delta\mu_{\min}$ between $\bar{x} = 1/5$ and annihilating case.

5.3 Maximizing force advantage

Here we return to $\bar{G} = \bar{I} = 0$, the war of annihilation. Suppose that, rather than the previous problem of choosing μ to minimize the cost of a marginal win, we instead choose μ to maximize the firepower advantage (the difference between left- and right-hand sides of (8)) available at a given cost $C = c_1 \gamma G_0^2 + c_2 \mu$.

Thus we need to find the value of μ which maximizes

$$\frac{C - c_2\mu}{c_1} - \frac{2\alpha P^2}{1 - \mu}\left[x - \frac{\mu}{1 - \mu}\log\left(1 + \frac{1 - \mu}{\mu}x\right)\right],$$

or (equivalently) which minimizes

$$2\alpha P^2 f_{x,r}(\mu).$$

But this is precisely the same problem, with the same solution, as in sub-section 5.1. It makes no difference whether the state wishes to minimize the cost of a bare win, or to maximize its force advantage (and thereby minimize its losses) for a given cost outlay. Either way, the optimum level of intel is the same.

6 Conclusions and Operational Lessons

We have written down and analyzed a variant of Lanchester's models that mixes aimed and unaimed fire by linear interpolation. This allowed us to model, in the simplest possible setting, the trade-off between targeting and firing rate which is a crucial component of the operational use of many weapons systems.

As an application, we examined this trade-off in the context of counterinsurgent warfare, in which state forces target an insurgency with a mix of aimed (well-targeted, 'intelligent') and unaimed (random, 'unintelligent') fire. In contrast to other recent dynamical-systems models of insurgent war (*e.g.* [7, 12]), we included no psychological variables, parameters or feedbacks, no dynamics of popular opinion, no game theory. This was a purely attritional model, with all the acknowledged deficiencies of these, in which the question was posed only at its simplest: which, for various parameter regimes, is more likely to lead to a state victory, 'bits'—better intelligence—or 'shots'—increased firepower?

Even in this context, of attrition and annihilation, in which no account was taken of the human (and, in the end, political) costs of random violence, intelligence emerges as remarkably valuable. First, whether from the point of view of minimizing cost or of maximizing force advantage (both in section 5), it is always the case that if intel is sufficiently cheap then a force can never have too much of it: perfect intel is the most effective option. This is true whether in a battle of annihilation or of limited loss toleration, although this 'sufficiently cheap' thresh-

old is reduced in the latter case. Estimates of realistic values of our cost ratio r would be difficult to justify because this type of data is typically classified. But, for a given cost ratio, it is clear that the optimal intel level increases rapidly as the accuracy of targeting x decreases.

Even without attempting a cost analysis, one can still compare intel with hit-rate and numbers (section 4). In our model, the state is fighting a square law battle, and a proportionate increase is twice as valuable in numbers as in hit-rate. But an (absolute) increase in intel is more valuable than a proportionate increase in (hit rate) $\times$ (numbers)^2 for most combinations of intel μ and accuracy x. This is our result at its starkest: bits are better than shots for all points below the curve in Fig. 2b, and this likely covers most realistic values of the parameters. Absent accurate estimates of these, and if the state has no strong reason to believe that its intel is already excellent (μ is close to 1) and its unaimed fire not too random (say $x > 1/2$), it should assume that more intel is the most cost-effective *military* option, independent of other considerations.

Finally, future work on dynamic combat models should focus more on psychological and social effects such as modeling the 'bandwagon' effect, which captures how people are mobilized to support or oppose a certain side in the conflict, in the presence of media-controlled public information. Also, as mentioned in Section 2, the case where the two sides face the bits-versus-shots dilemma naturally leads to a game-theoretic setting that may also be a subject for future research.

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